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Translated by L. K.

UDC 533. 93

SHAPING OF A TUBULAR BEAM OF CHARGED PARTICLES EMITTED WITH NONZERO VELOCITY AND IN A NONZERO FIELD STRENGTH

PMM Vol. 38, №1, 1974, pp. 78 - 83
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(Received March 23, 1973)

Within the framework of the hydrodynamic theory of dense beams of charged particles the problem of shaping of a tubular cylindrical flow has been solved. For emission limited by space charge and temperature, shaping electrodes have been constructed; they were computed for exact solution and for asymptotic expansion, equally valid near the flow boundary. The possibility of generalizing the proposed algorithms for the case of curvilinear trajectories is discussed.

In the hydrodynamic theory of dense beams the model of an emitter in the full space charge mode is most widely used. In this case for velocity U and field E, the zero values are taken. These assumptions lead to a fully defined form of the potential φ near the starting surface: in the flow domain $\varphi \sim z^{4/3}$ and in the Laplace domain $\varphi \sim \operatorname{Re}[z + i(R-R_0)]^{4/3}$. As the result we obtain a system of Pierce electrodes with the zero equipotential inclined to the beam boundary at the characteristic initial angle of 67.5°, irrespective of the emitter curvilinearity and of the density of current J at it.

However, in recent times the interest has increased for triode guns with grid control [1 — 4], guns giving sharp deceleration of the flow [5] and guns with autoemissive cathodes. These structures are distinguished by a more complicated singularity for the potential in the low velocity domain. This singularity for a unidimensional flow between parallel planes are given by the following parametric equations:

$$z = {}^1/_6\, Jt^3 + {}^1/_2 Et^2 + Ut, \quad v_z = dz / dt, \quad 2\varphi = v_z^2$$

The specific charge $\eta = e / m$ is omitted for reasons of convenience; the change $\eta\varphi \to \varphi$, $4\pi\eta J \to J$, $z = 0$, $t = 0$ correspond to the emitter.

For the structures mentioned above it is necessary to compute the shaping electrodes for the domain where the terms proportional to J, E and U are commensurable. In the triode gun, such a situation occurs when the potential of the control grid deviates from its inherent value, i.e. from the value defined by the $^3/_2$ power law. In this case the grid can be considered as an emitter with nonzero conditions on it.

The shaping of a tubular cylindrical beam has been the problem which has attracted the attention of several authors [6 – 8], till in [9] its closed solution in integral form was given. In [6 – 8] there are expansions which do not take into account the singularity in the full spatial charge mode and become divergent as the emitter is approached.

The computation problem for narrow tubular beams under nonzero conditions at the emitter is solved below by means of asymptotic expansions. The same solution also permits a description of the case of sharp deceleration. An exact solution of the shaping problem of a cylindrical beam for different modes of emission is given. The results obtained by asymptotic expansions are compared with the exact expressions.

1. Construction of the solution in the form of an asymptotic series. Let R, z be cylindrical and s, l – dimensionless coordinates connected by the relations

$$R = R_* (1 + \mu s), \qquad z = a_* l$$

Here R_* is the characteristic beam radius (internal or external), a_* is the characteristic strip width, μ is the symbol indicating that s is the value of the first order smallness in comparison with unity. Using the variables l, s the Laplace equation has the form

$$\frac{\partial^2\varphi}{\partial l^2} + \frac{\partial^2\varphi}{\partial s^2} = -\frac{\mu}{1+\mu s}\frac{\partial\varphi}{\partial s} \tag{1.1}$$

We have to find the solution of (1.1) satisfying the following conditions along the boundary $s = 0$

$$2\varphi\,|_{s=0} = \Phi(t) = ({}^1\!/_2 J t^2 + Et + U)^2, \qquad \partial\varphi / \partial s|_{s=0} = 0 \tag{1.2}$$
$$l = {}^1\!/_6 J t^3 + {}^1\!/_2 E t^2 + Ut$$

The formulas (1.2) are valid both for temperature limited emission ($E \neq 0,\ U = 0$) and for the full space charge mode ($E = 0,\ U = 0$). We perform the conformal transformation

$$l + is = {}^1\!/_6 J w^3 + {}^1\!/_2 E w^2 + Uw = F(w)$$

and express Eq. (1.1) in the coordinates $w = u + iv,\ \overline{w} = u - iv$ (hereafter the dash denotes the complex conjugate)

$$\frac{\partial^2\varphi}{\partial w\,\partial\overline{w}} = \frac{1}{2}\,\frac{\mu}{1+\mu(F-\overline{F})/2i}\,\frac{1}{2i}\left(\overline{F}_w\frac{\partial\varphi}{\partial w} - F_w\frac{\partial\varphi}{\partial\overline{w}}\right), \qquad F_w = \frac{dF}{dw}$$

Expressing the potential in the form

$$2\varphi = \sum_{n=0}^{\infty}\varphi\langle n\rangle\mu^n, \qquad \varphi_n = \sum_{k=0}^{n}\varphi\langle k\rangle\mu^k$$

for the n-th term, we obtain

$$\frac{\partial^2\varphi\langle n\rangle}{\partial w\,\partial\overline{w}} = \frac{1}{2}\sum_{k=0}^{n-1}(-1)^k\left(\frac{F-\overline{F}}{2i}\right)^k\frac{1}{2i}\left[\overline{F}_w\frac{\partial\varphi\langle n-k-1\rangle}{\partial w} - \right. \tag{1.3}$$

$$F_w \frac{\partial \varphi \langle n-k-1 \rangle}{\partial \overline{w}} \Big] = f_n(w, \overline{w})$$

The solution of the problem (1.1), (1.2) is now given by the formulas

$$\varphi \langle 0 \rangle = \frac{1}{2}(\Phi + \overline{\Phi}), \qquad \varphi \langle n \rangle = \int_{\overline{w}}^{w} dx \int_{x}^{\overline{w}} f_n(x, y)\, dy \tag{1.4}$$

The first few terms of the expansion computed using (1.3), (1.4) are given below

$$\varphi\langle 0 \rangle = \mathrm{Re}[{}^1/_4 J^2 w^4 + JEw^3 + (E^2 + JU)w^2 + 2EUw] \tag{1.5}$$

$$\begin{aligned}\varphi\langle 1 \rangle = &-{}^1/_{12}\mathrm{Im}\,[{}^1/_{56} J^3 w^7 (1 - 7\Psi^3) + {}^1/_8 J^2 E w^6 (1 - 3\Psi^2) + \\ &{}^1/_5 JE^2 w^5 (1 - 5\Psi^2) + {}^7/_{20} J^2 U w^5 (1 + {}^{10}/_7 \Psi^2 + {}^{15}/_7 \Psi^4) + \\ &JEUw^4 (1 + 2\Psi^3) + JU^2 w^3 (1 + 3\Psi^2)]\end{aligned}$$

$$\begin{aligned}\varphi\langle 2 \rangle = &-{}^1/_{64}\mathrm{Re}\,[{}^1/_{168} J^4 w^{10} (1 - 8\Psi^3 - 7\Psi^4) + {}^5/_{84} J^3 E w^9 (1 - \\ &{}^{12}/_5 \Psi^2 - {}^{14}/_{15} \Psi^3 + {}^{21}/_5 \Psi^4) + {}^{23}/_{120} J^2 E^2 w^8 (1 - {}^{100}/_{23} \Psi^2 + \\ &{}^{32}/_{23} \Psi^3 + {}^{45}/_{23} \Psi^4) + {}^1/_5 JE^3 w^7 (1 - 6\Psi^2 + 5\Psi^3) + \\ &{}^{103}/_{2520} J^3 U w^8 (1 - {}^{180}/_{103} \Psi + {}^{280}/_{103} \Psi^2 - {}^{1148}/_{103} \Psi^3 + \\ &{}^{945}/_{103} \Psi^4) + {}^{203}/_{40} J^2 EUw^7 (1 - {}^{220}/_{261} \Psi - {}^{28}/_{29} \Psi^2 + {}^{140}/_{261} \Psi^3) + \\ &{}^{56}/_{15} JE^2 U w^6 (1 - {}^{17}/_7 \Psi + {}^5/_7 \Psi^2 + {}^5/_7 \Psi^3) + {}^{31}/_{45} J^2 U^2 w^6 (1 - \\ &{}^{96}/_{31} \Psi + {}^{105}/_{31} \Psi^2 - {}^{10}/_{31} \Psi^3) - {}^2/_3 E^3 U w^5 (1 - 3\Psi + 2\Psi^2) + \\ &2JEU^2 w^5 (1 - 3\Psi + 2\Psi^2) + {}^4/_3 JU^3 w^4 (1 - 4\Psi + 3\Psi^2)]\end{aligned}$$

$$\Psi = \overline{w} / w$$

It is obvious, that $\varphi\langle 0 \rangle$ represents electrodes obtained by the rotation around z of equipotential curves from the two-dimensional problem and subsequent approximations give corrections under axially symmetric conditions.

2. Exact solution of the shaping problem of a cylindrical beam. It was mentioned above that the solution of this problem in integral form was quoted in [9]. However it is more convenient to represent the Riemann function not by an elliptical integral but by a hypergeometric function

$$G = F({}^1/_2, {}^1/_2; 1; \lambda), \quad \lambda = -[(R - R_c)^2 + (z - z_c)^2] / 4RR_c \tag{2.1}$$

and to use for the potential the formula following from such a representation under Cauchy conditions, defined by the parametric expressions (1.2). Here R_c, z_c are the coordinates of the observation point at which the potential is computed. Then applying analytical continuity of the Cauchy conditions the following expressions result:

$$R \to R_e, z \to z_e, \quad R_c \to R, z_c \to z$$

The parametric equations of a cylindrical boundary have the form

$$R_e = 1, \quad z_e = {}^1/_6\, t^3 + {}^1/_2 \gamma t^2 + \delta t, \quad \gamma = E \,/\, J, \quad \delta = U \,/\, J \tag{2.2}$$
$$\alpha = dR_e \,/\, dt = 0, \quad \beta = dz_e \,/\, dt = {}^1/_2 t^2 + \gamma t + \delta$$
$$\varphi \,|_{R=1} = \Phi(t) = ({}^1/_2 t^2 + \gamma t + \delta)^2$$

The expressions (2.2) are given in dimensionless variables, the meaning of which easily results from (1.2) and which will be used in the further examination and comparison with the asymptotic series. Finally we obtain the following form for the potential

$$2\varphi(u, v) = \frac{1}{\sqrt{R}} \operatorname{Re} \left\{ \Phi(w) + \frac{1}{2} \int_0^v \left[F\left(\frac{1}{2}, \frac{1}{2}; 1; \lambda_e\right) - \frac{R^2 - 1 + [z_e(\zeta) - z]^2}{8R} F\left(\frac{3}{2}, \frac{3}{2}; 2; \lambda_e\right) \right] \Phi^{3/2}(\zeta)\, d\xi \right\} \tag{2.3}$$
$$\lambda_e = -\frac{(1 - R)^2 + [z_e(\zeta) - z]^2}{4R}, \quad w = u + iv, \quad \zeta = u + i\xi$$

For the case in which the distance from the boundary is large, a different form of the hypergeometric functions may be more useful

$$F({}^1/_2, {}^1/_2; 1; \lambda_e) = (1 - \lambda_e)^{-1/2} F({}^1/_2, {}^1/_2; 1, \Lambda_e)$$
$$F({}^3/_2, {}^3/_2; 2; \lambda_e) = (1 - \lambda_e)^{-3/2} F({}^3/_2, {}^1/_2; 2; \Lambda_e)$$
$$\Lambda_e = \lambda_e \,/\, (\lambda_e - 1) = [(1 - R)^2 + (z_e - z)^2] \,/\, [(1 + R)^2 + (z_e - z)^2]$$

and the corresponding expression for the potential

$$2\varphi(u, v) = \operatorname{Re} \left\{ \frac{1}{\sqrt{R}} \Phi(w) + \int_0^v \left[F\left(\frac{1}{2}, \frac{1}{2}; 1; \Lambda_e\right) - \frac{1}{2} \frac{R^2 - 1 + (z_e - z)^2}{(1 + R)^2 + (z_e - z)^2} F\left(\frac{3}{2}, \frac{1}{2}; 2; \Lambda_e\right) \right] \frac{\Phi^{3/2}(\zeta)\, d\xi}{[(1 + R)^2 + (z_e - z)^2]^{1/2}} \right\} \tag{2.4}$$

If in a hypergeometric function $F(a, b; c; z)$ the modulus of z exceeds unity or is close to it, then it is necessary to use the formulas for analytical continuity quoted in [10].

The construction of the equipotential lines $v = v(u, \varphi)$ is reduced to finding the root of Eq. (2.3) or (2.4) for fixed values u and φ. Transformation to R, z is made according to the formulas

$$z + i(R - 1) = {}^1/_6\, w^3 + {}^1/_2 \gamma w^2 + \delta w \tag{2.5}$$

In solving Eqs. (2.3), (2.4), it is useful to imagine a pattern of surfaces $v = v(u, \varphi)$, the qualitative behavior of which in the meridional plane is known [7, 11]. In (R, z) the part of the first quadrant where $\varphi \geqslant 0$, is of interest. Since (2.5) maps $R = 1$ as $v = 0$, it is sufficient to investigate the ray $z = 0$ for $R \geqslant 0$. With $\delta \neq 0$ near the origin of the coordinates, the pattern of equipotential curves changes little, whereas for very large $|w|$ the main term in (2.5) is a cubic one and reduces the angles by a factor of three. With $\delta = 0$, $\gamma \neq 0$ and small $|w|$, the angles are reduced by half and for a ray undergoing a further transformation its inclination becomes still smaller.

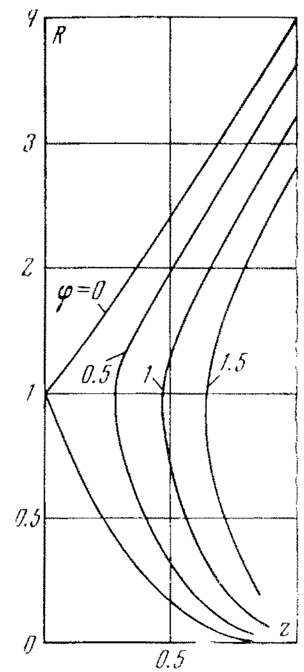

Fig. 1

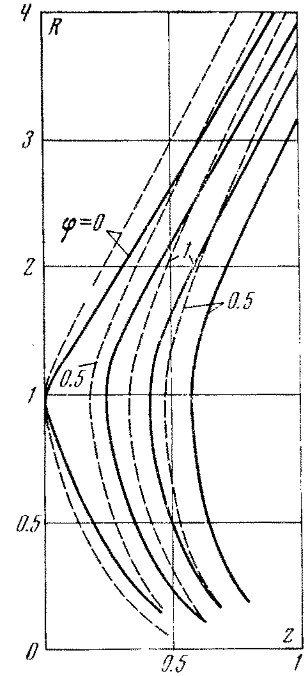

Fig. 2

In Fig. 1 the curves at $\varphi = \text{const}$ are shown for the case of full space charge ($\gamma = 0$), in Fig. 2 — for temperature limited emission ($\gamma = 0.5$ — solid curves and $\gamma = 1$ — dotted curves). It is obvious that an increase in γ causes a contraction of the pattern of the equipotential surfaces. Fig. 3 shows clearly the behavior of the curves $\varphi = \text{const}$ with nonmonotonic behavior of the potential for $\varphi_{\min} = {}^1/_5$ (solid lines) and $\varphi_{\min} = {}^4/_5$ (dotted lines).

3. Discussion of results. The exact solution of the problem of shaping a tubular beam allows to establish the domain of applicability of the asymptotic expansions. On Fig. 4 for comparison the curves $\varphi = 0$ computed using (2.3), (2.4) and those obtained using the asymptotic series at $\gamma = 0$ are shown.

For $R > 1$ the convergence of φ_n towards an exact solution by approximations has an oscillatory character. We not that the series for $1 / R$ used in the construction of the asymptotic expansions diverges at $R \geqslant 2$. Starting with $R = 2.3$, $\varphi \langle 3 \rangle$ causes a considerable error and soon becomes useless. For $R < 1$ a monotonic convergence towards an exact curve takes place, while the difference in the coordinates of the zero potential line becomes noticeable at $R < 0.2$ (7% for φ_3 and 15% for φ_2 at $R = 0.1$). For shaping electrodes with $\varphi \neq 0$ and also at $\gamma \neq 0$ the approximate solution behaves analogously both qualitatively and quantitatively. It is interesting to note that for $\gamma = 0$ to 10 the curves computed on the basis of φ_1, φ_2 from the formula

$$z = {}^1/_7\ [4f_1\ (R,\ \varphi) + 3f_2\ (R,\ \varphi)],\qquad R \geqslant 1$$

are practically identical to the family for $\varphi = \text{const}$ defined by the formulas (2.3), (2.4). Here $z = f_1\ (R,\ \varphi)$, $z = f_2\ (R,\ \varphi)$ are the equipotential lines for the first and second approximations.

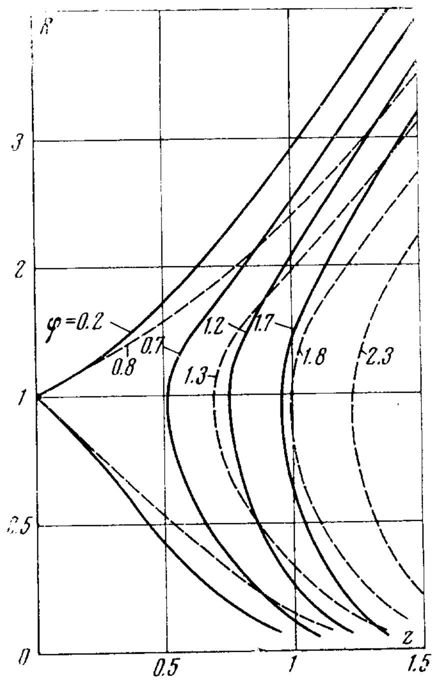

Fig. 3

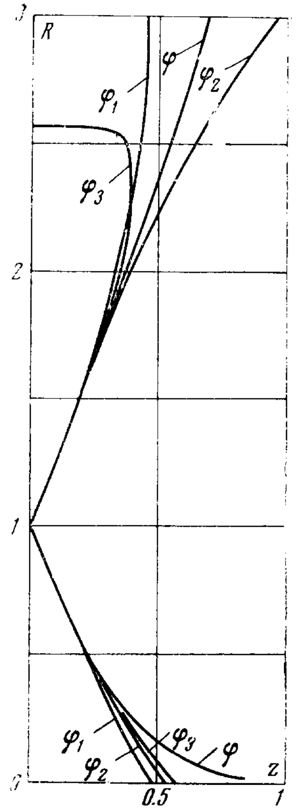

Fig. 4

In this way, asymptotic expansions can be successfully used not only for the computation of guns giving thin walled tubular beams but also for solid flows.

In curvilinear beams near the emitter (and the collector in the case of a sharp deceleration) the structure of a singularity is the same as that for a straight-line current. Within a paraxial approximation the expansion for the potential in coordinates corresponding to the trajectories can be expressed as the product of the square of the velocity V along the axis containing the whole singularity by a regular function having the form of a power series in the distance from the beam axis [12]. For an examination of the external problem of a gun with a curvilinear tubular beam, it is sufficient to determine the dependences $z\ (t)$, $V\ (t)$ and to solve the Laplace equation for more general Cauchy conditions on a flow surface, using the method of constructing an asymptotic series described above.

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Translated by H.B.